Alternating Series:

One of our main objectives is to be able to represent complicated functions as convergent power series (infinite polynomials). For example, it turns out that

$$\text{for each } x,\ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots + \frac{x^{2n+1}}{(2n+1)!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Note: From this we will obtain

$$\frac{d}{dx} \sin x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

When do alternating series converge?

Theorem: The Alternating Series Test (Leibniz's Test)

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive terms such that

i. $a_{n+1} \leq a_n$ for all $n \geq N$ (N a natural number) and

ii. $\lim_{n \to \infty} a_n = 0$

then the series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$ converges. (In words, if the terms of an alternating series decrease monotonically to 0 in absolute value, then it converges. And as with many other tests, the result is still true if condition (i) is only true for all $n \geq N$ for some natural number N.)

Example 1. The alternating harmonic series is given below.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

By the alternating series test this series converges but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Example 2.

Determine whether the alternating series $\sum_{n=2}^{\infty}(-1)^n \frac{3}{n\ln n}$ converges.

$\lim_{n\to\infty} \frac{3}{n \ln n} = 0$ and $\frac{3}{(n+1)\ln(n+1)} \leq \frac{3}{n \ln n}$

for each n. By the alternating series test

the series converges.

Example 3.

Approximate the sum of the series $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{2}{n^4}$

accurate to two decimal places.

We know $|r_n| = |s - s_n| \leq a_{n+1}$

If $a_{n+1} = \frac{2}{(n+1)^4} < .009$ then s_n, the

n-th partial sum is accurate to 2

decimal places. So $(n+1)^4 > \frac{2}{.009}$,

$n+1 > 3.86$, $n > 2.86$, $n \geq 3$

$s_3 = 2 - \frac{2}{2^4} + \frac{2}{3^5} = 1.88$

Definition: Absolute and Conditional Convergence

We say that a series $\sum a_n$ **converges absolutely** (or is **absolutely convergent**) if the corresponding series $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ does not converge, we say $\sum a_n$ **converges conditionally** (or is **conditionally convergent**).

Example 4.

Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{7}{n^2+9}$ converges absolutely, converges conditionally, or diverges.

Since $\sum_{n=1}^{\infty} \left| (-1)^{n+1} \frac{7}{n^2+9} \right|$

$$= \sum_{n=1}^{\infty} \frac{7}{n^2+9} \leq 7 \cdot \sum_{n=1}^{\infty} \frac{1}{n^2} \quad +$$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series is absolutely convergent.

Table 1: Summary of Convergence Tests

Name of Test	Form of Series	Convergence Condition	Divergence Condition	Notes
n^{th}-Term Divergence Test	$\sum_{n=1}^{\infty} a_n$	Does not apply	$\lim_{n\to\infty} a_n \neq 0$	Only used to prove divergence
Geometric Series	$\sum_{n=1}^{\infty} ar^{n-1}$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ when $\lvert r\rvert < 1$
Integral Test	$\sum_{n=N}^{\infty} a_n$, $a_n = f(n)$ where f is a continuous, positive, and decreasing function for $n \geq N$	$\int_N^{\infty} f(x)dx < \infty$	$\int_N^{\infty} f(x)dx = \infty$	The value of the integral is not the value of the sum.
p-Series	$\sum_{n=1}^{\infty} \left(\frac{1}{n^p}\right)$	$p > 1$	$p \leq 1$	Harmonic series corresponds to $p = 1$
Direct Comparison Test	$\sum_{n=1}^{\infty} a_n$, $a_n \geq 0$	$a_n \leq b_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ converges	$0 \leq b_n \leq a_n$ for all $n \geq N$ and $\sum_{n=1}^{\infty} b_n$ diverges	Finding a good comparison series $\sum_{n=1}^{\infty} b_n$ is the key.
Limit Comparison Test	$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$, $a_n > 0$ and $b_n > 0$	$0 \leq \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$ and $\sum_{n=1}^{\infty} b_n$ converges	$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) > 0$ or $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges	Finding a good comparison series $\sum_{n=1}^{\infty} b_n$ is the key.
Ratio Test	$\sum_{n=1}^{\infty} a_n$ $a_n > 0$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) > 1$ or $\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \infty$	Inconclusive if $\lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = 1$
Root Test	$\sum_{n=1}^{\infty} a_n$ $a_n > 0$	$\lim_{n\to\infty} \sqrt[n]{a_n} < 1$	$\lim_{n\to\infty} \sqrt[n]{a_n} > 1$ or $\lim_{n\to\infty} \sqrt[n]{a_n} = \infty$	Inconclusive if $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$
Alternating Series Test	$\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ $a_n > 0$ and $a_{n+1} \leq a_n$	$\lim_{n\to\infty} a_n = 0$	$\lim_{n\to\infty} a_n \neq 0$	n^{th} remainder $\lvert r_n\rvert \leq a_{n+1}$